1. State true or false. Justify your answers. No marks will be awarded in absence of proper justification.

1(a) 0 is not an eigenvalue of the differential operator $D : P_n(\mathbb{R}) \longrightarrow P_n(\mathbb{R})$, where $P_n(\mathbb{R})$ is the space of all real polynomials of degree less than or equal to n.

Ans. False. For any nonzero constant polynomial c we have $D(c) = 0 = 0.c$ and hence 0 is an eigenvalue of D.

1(b) There exist invertible skew-symmetric real matrices of odd order.

Ans. False. Let A be skew-symmetric real matrices of odd order n, i.e, $A^t = -A = -I_n \cdot A$, where I_n is the indentity matrix of order n. Hence, $det(A) = det(A^t) = det(-I_n) \cdot det(A) = (-1)^n det(A) = -det(A)$. We get $2 \cdot det(A) = 0$ which implies $det(A) = 0$ and A is not invertible.

1(c) For a field $\mathbb{F}$, bases of vector spaces $\mathbb{F}^n$ are in bijective correspondence with elements of $GL_n(\mathbb{F})$.

Ans. True. Let S be the set of all bases of $\mathbb{F}^n$ and let $B \in S$. Let $B = \{b_1, \ldots, b_n\}$, where each b_i is a column vector of length n. We define a map $f : S \longrightarrow GL_n(\mathbb{F})$ such that $f(B) = (b_1 \cdots b_n)$, a $n \times n$ matrix with b_i^s are the columns. This f gives a bijection between S and $GL_n(\mathbb{F})$.

If

2.(a) Find a basis for the null space of the matrix

$$A = \begin{pmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & 2 \\ 0 & 2 & -8 & 1 & 9 \end{pmatrix}.$$

Ans. Reduced row echelon form of the above matrix is

$$\begin{pmatrix} 1 & 0 & -5 & 0 & 15 \\ 0 & 1 & -4 & 0 & 10 \\ 0 & 0 & 0 & 1 & -11 \end{pmatrix}$$

and hence rank$(A) = 3$. Therefore, dimension of the null space of A will be 2 (as rank(A)+nullity$(A) = 5$). Let (u, v, w, x, y) be an element of the null space. From the equation $A(u, v, w, x, y,)^t = 0$ we get

$$\begin{aligned} u - 5w + 15y &= 0 \\ v - 4w + 10y &= 0 \\ x - 11y &= 0. \end{aligned}$$

Therefore, $u = 5w - 15y, v = 4w - 10y, x = 11y, w$ and y are arbitrary. Hence a basis of the null space consists of $(5, 4, 1, 0, 0)^t$ and $(-15, -10, 0, 11, 1)^t$.

2(b) Let A be the $n \times n$ backward identity matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

What is determinant of A? Find A^{-1}.

Ans. $\det(A) = 1$ if $n = 0$ or $1 \bmod(4)$. And $\det(A) = -1$ if $n = 2$ or $3 \bmod(4)$. Note that $A^2 = id$ and hence $A^{-1} = A$.

3. Define $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by $T(x, y, z)^t = (x + y + 2z, x - y - 3z, 2x + 3y + z)^t$. Let $\mathcal{B}_1$ be the standard basis and $\mathcal{B}_2 = \{(1, 1, 1)^t, (1, -1, 1)^t, (1, 1, 2)^t\}$ be another ordered basis of $\mathbb{R}^3$.

3(a) Find the matrix of T with respect to $\mathcal{B}_1$, say A_1.

Ans. Let $\mathcal{B}_1 = \{v_1, v_2, v_3\} = \{(1, 0, 0)^t, (0, 1, 0)^t, (0, 0, 1)^t\}$. Note that $T(v_1) = (1, 1, 2)^t = v_1 + v_2 + 2v_3, T(v_2) = (1, -1, 3)^t = v_1 - v_2 + 3v_3, T(v_3) = (2, -3, 1)^t = 2v_1 - 3v_2 + v_3$. Therefore,

$$A_1 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 3 \\ 2 & -3 & 1 \end{pmatrix}$$

3(b) Find the matrix of T with respect to $\mathcal{B}_2$, say A_2.

Ans. Let $\mathcal{B}_2 = \{w_1, w_2, w_3\} = \{(1, 1, 1)^t, (1, -1, 1)^t, (1, 1, 2)^t\}$. Note that $T(w_1) = (4, -3, 6)^t = -(3/2)w_1 + (7/2)w_2 + 2w_3, T(w_2) = (2, -1, 0)^t = (5/2)w_1 + (3/2)w_2 - 2w_3, T(w_3) = (6, -6, 7)^t = -w_1 + 6w_2 + w_3$. Therefore,

$$A_2 = \begin{pmatrix} -3/2 & 7/2 & 2 \\ 5/2 & 3/2 & -2 \\ -1 & 6 & 1 \end{pmatrix}$$

3(c) Find the matrix P such that $PA_1P^{-1} = A_2$.

Ans. The matrix P will look like

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

To know how to construct the matrix P see *Topics in Algebra* by I.N. Herstein. See chapter 6, section 3.

4.(a) Show that A and A^t have the same set of eigenvalues. Give examples to show that eigen vectors of A and A^t may be different.

Ans. Let λ be an eigenvalue of A. That means $A - \lambda I$ is not invertible, i.e, $\det(A - \lambda I)$ is not a unit. Note that $\det(A - \lambda I) = \det(A - \lambda I)^t = \det(A^t - \lambda I^t) = \det(A^t - \lambda I)$, and hence λ is an eigenvalue of A^t.

4(b) Let $A = (a_{ij})$ be an $n \times n$ matrix. Suppose that for all i, $1 \leq i \leq n, \sum_{j=1}^{n} a_{ij} = 1$. Then prove that 1 is an eigenvalue of A. What is the corresponding eigenvector?

Ans. Consider the column vector $v = (1, 1, \ldots, 1)^t$ of length n. Note that $A.v = 1.v$, and hence 1 is an eigenvalue of A. This shows v is the corresponding eigenvector.

4(c) Suppose all the column sums of B are equal to 1. Does the same result hold?

Ans. In 4(a) we have seen that A and A^t have the same set of eigenvalues. Therefore, the same result holds for B.

6(a) Define row rank and column rank of a matrix.

Ans. Let A be an $m \times n$ matrix with entries from a field $\mathbb{F}$, i.e, A has m rows each is of length n. These m rows of A will generate a subspace of $\mathbb{F}^n$. The dimension of this subspace is called the row rank on A. Note that A has n columns each is of length m. These n columns of A will generate a subspace of $\mathbb{F}^m$. The dimension of this subspace is called the column rank of A.

6(b) Show that row rank of a matrix is equal to its column rank.

Ans. See the following link for a complete solution.
www.mtts.org.in/userapps/download-expo.php?fileid=64

6(c) Let A be a $m \times n$ matrix and B be a $n \times k$ matrix. Prove that $rank(AB) \leq min\{rank(A), rank(B)\}$.

Ans. The link for the previous solution also contains a solution for 6(c).

Alternative solution: A $m \times n$ matrix can be treated as a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^m$. In that case rank of the matrix is the dimension of the image space of the transformation. If T is a linear transformation from V_1 to V_2 then clearly $dim\ T(V_1) \leq dim(V_1)$. Hence $rank(AB) \leq rank(B)$. Since row rank and column rank of a matrix are equal, we have

$$rank(AB) = rank(AB)^t = rank(B^t A^t) \leq rank(A^t) = rank(A)$$

Therefore, $rank(AB) \leq min\{rank(A), rank(B)\}$.